Qregularity and tensor products of vector bundles on smooth quadric hypersurfaces

Edoardo Ballico and Francesco Malaspina

Università di Trento
38050 Povo (TN), Italy
e-mail: ballico@science.unitn.it

Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: francesco.malaspina@polito.it

Abstract

Let $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. Here we prove that the tensor product of an m-Qregular sheaf on $\mathcal{Q}_n$ and an l-Qregular vector bundle on $\mathcal{Q}_n$ is $(m+l)$-Qregular.

1 Introduction

Let $\mathcal{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. We use the unified notation Σ_* meaning that for even n both the spinor bundles Σ_1 and Σ_2 are considered, while $\Sigma_* = \Sigma$ if n is odd. We recall the definition of Qregularity for a coherent sheaf on $\mathcal{Q}_n$ given in [2]:

Definition 1.1. A coherent sheaf F on $\mathcal{Q}_n$ $(n \geq 2)$ is said to be m-Qregular if one of the following equivalent conditions are satisfied:

1. $H^i(F(m-i)) = 0$ for $i = 1, \ldots, n-1$, and $H^n(F(m) \otimes \Sigma_*(-n)) = 0$.

2. $H^i(F(m-i)) = 0$ for $i = 1, \ldots, n-1$, $H^{n-1}(F(m) \otimes \Sigma_*(-n+1)) = 0$, and $H^n(F(m-n+1)) = 0$.

In [2] we defined the Qregularity of F, $Qreg(F)$, as the least integer m such that F is m-Qregular. We set $Qreg(F) = -\infty$ if there is no such an integer.

Here we prove the following property of Qregularity.

Theorem 1.2. *Let F and G be m-Qregular and l-Qregular coherent sheaves such that $Tor_i(F, G) = 0$ for $i > 0$. Then $F \otimes G$ is $(m+l)$-Qregular. In particular this holds if one of them is locally free.*

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The corresponding result is true taking as regularity either the Castelnuovo-Mumford regularity or (for sheaves on a Grassmannian) the Grassmann regularity defined by J. V. Chipalkatti ([3], Theorem 1.9). The corresponding result is not true (not even if G is a line bundle) on many varieties with respect to geometric collections or n-block collections (very general and very important definitions of regularity discovered by L. Costa and R.-M. Miró-Roig) ([4], [5], [6]). Our definition of Qregularity on smooth quadric hypersurfaces was taylor-made to get splitting theorems and to be well-behaved with respect to smooth hyperplane sections. Theorem 1.2 gives another good property of it. To get Theorem 1.2 we easily adapt Chicalpatti's proof of [3], Theorem 1.9, except that we found that in our set-up we need one more vanishing. Our proof of this vanishing shows that on smooth quadric hypersurfaces our definition of Qregularity easily gives splitting results (see Lemma 2.2).

2 The proof

Set $\mathcal{O} := \mathcal{O}_{\mathcal{Q}_n}$.

Lemma 2.1. *Let F be a 0-Qregular coherent sheaf on $\mathcal{Q}_n$. Then F admits a finite locally free resolution of the form:*

$$0 \to K^n \to \cdots \to K^0 \to F \to 0,$$

where K^j $(0 \leq j < n)$ is a finite direct sum of line bundles $\mathcal{O}(-j)$ and K^n is an n-Qregular locally free sheaf.

Proof. Since F is globally generated ([2], proposition 2.5), there is a surjective map

$$H^0(F) \otimes \mathcal{O} \to F.$$

The kernel K is a coherent sheaf and we have the exact sequence

$$0 \to K \to H^0(F) \otimes \mathcal{O} \to F \to 0.$$

Since the evaluation map $H^0(F) \otimes \mathcal{O} \to F \to 0$ induces a bijection of global sections, $H^1(K) = 0$. From the sequences

$$H^{i-1}(F(-i+1)) \to H^i(K(-i+1)) \to H^0(F) \otimes H^i(\mathcal{O}(-i+1)) \to 0,$$

we see that $H^i(K(-i+1)) = 0$ for any i $(1 < i < n)$.
From the sequences

$$H^{n-1}(F) \otimes \Sigma_*(-n+1) \to H^n(K(1) \otimes \Sigma_*(-n)) \to H^0(F) \otimes H^n(\Sigma_*(-n+1)) \to 0,$$

we see that $H^n(K(1) \otimes \Sigma_*(-n)) = 0$. We conclude that K is 1-Qregular.
We apply the same argument to K and we obtain a surjective map

$$H^0(K(1)) \otimes \mathcal{O}(-1) \to K$$

with a 2-Qregular kernel. By the syzygies Theorem we obtain the claimed resolution.

□

Lemma 2.2. *Let G an m-Qregular coherent sheaf on $\mathcal{Q}_n$ such that $h^n(G(-m-n)) \neq 0$. Then G has $\mathcal{O}(-m)$ as a direct factor.*

Proof. Since $h^n(G(-m-n)) \neq 0$, $h^0(G^*(m)) \neq 0$ ([1], theorem at page 1). Hence there is a non-zero map $\tau : G(m) \to \mathcal{O}$. Since $G(m)$ is 0-Qregular, it is spanned ([2], proposition 2.5), i.e. there are an integer $N > 0$ and a surjection $u : \mathcal{O}^N \to G(m)$. Every non-zero map $\mathcal{O} \to \mathcal{O}$ is an isomorphism. Hence $\tau \circ u$ is surjective and there is $v : \mathcal{O} \to \mathcal{O}^N$ such that $(\tau \circ u) \circ v$ is the identity map of $\mathcal{O}$. Hence the maps τ and $v \circ u : \mathcal{O} \to G(m)$ show that $G(m) \cong \mathcal{O} \oplus G'$ with $G' \cong \mathrm{Ker}(\tau)$. □

Proof of Theorem 1.2. We first reduce to the case in which G is indecomposable. Indeed, if $G \cong G_1 \oplus G_2$ where G_1 is l-Qregular and G_2 is l'-Qregular ($l' \leq l$), then $F \otimes G_1$ is $(l+m)$-Qregular and $F \otimes G_2$ is $(l'+m)$-Qregular ($l'+m \leq l+m$) so $F \otimes G \cong (F \otimes G_1) \oplus (F \otimes G_2)$ is $(l+m)$-Qregular.
We can assume that G is not $\mathcal{O}(-l)$, because the statement is obviously true in this case. Hence by Lemma 2.2 we may assume $H^n(G(l-n)) = 0$. Let us tensorize by $G(l)$ the resolution of $F(m)$. We obtain the following resolution of $F \otimes G$:

$$0 \to K^n \otimes G(l) \to \cdots \to K^0 \otimes G(l) \to F \otimes G(m+l) \to 0,$$

where K^j $(0 \leq j < n)$ is a finite direct sum of line bundles $\mathcal{O}(-j)$ and K^n is a n-Qregular locally free sheaf.
Since

$$H^n(G(l-n)) = \cdots = H^1(G(l-1)) = 0,$$

we have $H^1(F \otimes G(m+l-1)) = 0$.
Since

$$H^n(G(l-n)) = \cdots = H^2(G(l-2)) = 0,$$

we have $H^2(F \otimes G(m+l-2)) = 0$ and so on.
Moreover, $H^n(G(l) \otimes \Sigma_*(-n)) = 0$ implies $H^n(F \otimes G(m+l) \otimes \Sigma_*(-n)) = 0$. Thus $F \otimes G$ is $(m+l)$-Qregular. □

Proposition 2.3. *Let F and G be m-Qregular and l-Qregular vector bundles on $\mathcal{Q}_n$. If F is not $(m-1)$-Qregular and G is not $(l-1)$-Qregular then $F \otimes G$ is not $(m+l-1)$-Qregular. In particular $Qreg(F) = Qreg(G) = 0$ implies $Qreg(F \otimes G) = 0$.*

Proof. By the above argument we can prove the result just for F and G indecomposable. Let us assume that G is not $(l-1)$-Qregular. We can assume that G is not $\mathcal{O}(-l)$, because the statement is obviously true in this case. Hence by Lemma 2.2 we may assume $H^n(G(l-n)) = 0$.
If $H^i(G(l-i-1)) \neq 0$ for some i $(0 > i > n)$, and

$$H^{i+1}(G(l-1-i-1)) = \cdots = H^n(G(l-n)) = 0,$$

we have an injective map

$$H^i(G(l-i-1)) \to H^i(F \otimes G(m+l-i-1))$$

and so $H^i(F \otimes G(m+l-i-1)) \neq 0$. This means that $F \otimes G$ is not $(m+l-1)$-Qregular.
If $H^i(G(l-i-1)) = 0$ for any i $(0 > i > n)$ but $H^{n-1}(G \otimes \Sigma_*(-n)) = 0$ by [2] Proof of Theorem 1.2., we have that $G \cong \Sigma_*(-l)$. By a symmetric argument we may assume that $F \cong \Sigma_*(-m)$. Now we only need to show that $\Sigma_*(-m) \otimes \Sigma_*(-l)$ is not $(m+l-1)$-Qregular. Indeed since $h^0(\Sigma_* \otimes \Sigma_*(-1)) = 0$, [2] Proposition 2.5 implies that $\Sigma_* \otimes \Sigma_*$ is not (-1)-Qregular.

□

Remark 2.4. On $\mathbb{P}^n$ if F is a regular coherent sheaf accoding Castelnuovo-Mumford, then it admits a finite locally free resolution of the form:

$$0 \to K^n \to \cdots \to K^0 \to F \to 0,$$

where K^j $(0 \leq j < n)$ is a finite direct sum of line bundles $\mathcal{O}(-j)$ and K^n is an n-regular locally free sheaf. Now arguing as above we can deduce that Theorem 1.2 and Proposition 2.3 hold also on $\mathbb{P}^n$ for Castelnuovo-Mumford regularity.

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